EXTENSIONS OF DEHN'S LEMMA AND THE LOOP THEOREM(1)

BY

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I. Introduction and Definitions

1. **Introduction.** This paper is devoted to extensions and strengthenings of the Loop Theorem and Dehn's Lemma. These two theorems were first proved by C. D. Papakyriakopoulos in [11] and [12] and have been immensely important in the recent development of 3-space topology.

The main technique used here is that of 2-sheeted coverings as developed by A. Shapiro and J. H. C. Whitehead in [14]. It will be assumed that the reader is familiar with §§1–3 of [14] and J. Stallings' proof of the Loop theorem in [15].

The problem involved is to take a given singular disk whose interior does not intersect the boundary and to "change" it into a nonsingular disk which has certain desired properties in common with the original singular disk. The primary difference between the methods used here and the previous work in the field is that we shall pay close attention to the geometrical relationships between the given singular disk and the obtained nonsingular disk. (See Theorem III.5.)

Theorem III.5 yields extensions of both the Loop Theorem (III.2) and of Dehn's Lemma (IV.3) and is used to prove the hitherto unpublished result (IV.2) that a tame simple closed curve in a 3-manifold, with or without boundary, bounds a disk if it can be shrunk to a point in its own complement. The extension of the Loop Theorem considers, in some cases, loops on 2-submanifolds of 3-manifolds.

The results of this paper are used in [8] to attack the general question proposed by R. H. Bing: *Does every simple closed curve which can be shrunk to a point in its own complement bound a disk*? As mentioned above the answer is yes, if the simple closed curve is tame. In [8] we give an affirmative answer in the case when the simple closed curve has finite penetration index and only a finite number of wild points, and also in the case that it has penetration index 2 at every point and the wild points form a tame 0-dimensional set. In addition, [8] gives a new condition for tameness of a simple closed curve.

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2. **Definitions and notations.**

I.1. Let K be a complex with a fixed triangulation. L is called a *subcomplex of* K if L is a complex each of whose simplexes is a simplex of K. L is called a *polyhedron in* K if L is a subcomplex of some subdivision of K.

I.2. If K is a complex and L a subset of $|K|$, then *the closed star of* L *in* K, $\text{st}(L,K)$, is the union of all (closed) simplexes of K which intersect L. *The open star of* L *in* K, $\text{ost}(L,K)$, is the union of all open simplexes whose closures intersect L.

I.3. A subset X of a complex K is called *tame* if there is a homeomorphism h of K onto K such that $h(X)$ is a polyhedron in K.

I.4. If L is a subcomplex of the subdivision αK of the complex K, then $\alpha_L^n K$ is called the *nth barycentric subdivision of* αK, *modulo* L and is defined inductively as follows: Let $A_1, A_2, A_3, \cdots$ be a well-ordering of the simplexes of αK not in L, such that $\dim(A_\delta) \leqq \dim(A_\gamma)$, whenever $\delta < \gamma$. (Note that for each γ, $L + \sum(A_\beta : 1 \leqq \beta < \gamma)$ is a subcomplex of αK.) We suppose that $\alpha_L^1[L + \sum(A_\beta : 1 \leqq \beta < \gamma)]$ has been defined, and we define $\alpha_L[L + \sum(A_\beta : 1 \leqq \beta < \gamma + 1)]$ to be $\alpha_L^1[L + \sum(A_\beta : 1 \leqq \beta < \gamma)]$ plus those simplexes which are equal to the cone, from the barycenter of A_γ, over a simplex of $(\text{Bd}\, A_\gamma) \cap (\alpha_L^1[L + \sum(A_\beta : 1 \leqq \beta < \gamma)])$. We define $\alpha_L^n K = \alpha_L^1(\alpha_L^{n-1} K)$.

Note that L is *a subcomplex of* $\alpha_L^n K$.

If $\emptyset$ *is the empty subcomplex, then* $\alpha^n K$ *denotes* $\alpha_\phi^n K$.

I.5. An *n-manifold* M^n is a separable metric space each of whose points has closed neighborhood homeomorphic to I^n.

The *boundary of* M^n, $\text{Bd}\, M^n$, is the set of points of M^n which do not have arbitrarily small neighborhoods homeomorphic to E^n.

A *combinatorial n-manifold* is a complex such that the closed star of each vertex has a rectilinear subdivision which is isomorphic to a rectilinear subdivision of an n-simplex.

It follows easily from [1], [10] that all 2- and 3-manifolds may be given combinatorial triangulations, and henceforth in this paper we shall assume that this has been done.

If N^p and M^n are triangulated manifolds of dimension p and n, respectively, then N^p is a *p-submanifold* of M^n if N^p is a subcomplex of M^n.

I.6. Let C^3 denote the solid cube in E^3 whose vertices are the eight points in E^3 which have as each coordinate either 1 or -1.

I.7. A mapping, f, from a complex K to a complex L is called *piecewise linear* (pwl) if the graph of f is a polyhedron in the product complex $K \times L$. If K is finite, then there are subdivisions α, β such that $f: \alpha K \to \beta L$ is simplicial.

I.8. If $f: K \to L$ is a pwl map of one complex into another, then by the *singularities of* f, $S(f)$, we shall mean the closure of the set of all points in L which are the images under f of more than one point of K. The cardinality of $f^{-1}(x)$ is called the *order* of x.

I.9. An *s-disk* (*loop, path*), D, in a manifold M is a mapping $D:\Delta \to M$, where Δ is the standard disk (simple closed curve, unit interval). D is called polyhedral if it is a pwl mapping. $D(\Delta)$ is sometimes written $|D|$ or, if no confusion is likely, D. $\mathrm{Bd}\,D = D|\mathrm{Bd}\,\Delta$ as maps, and $(D, \mathrm{Bd}\,D)$ is said to be in rnp in (M,N) if $D:(\Delta, \mathrm{Bd}\,\Delta) \to (M,N)$ is a rnp mapping (see I.13). An *arc* is a nonsingular path.

I.10. Let f be a mapping of a compact set A into a space B. Then f is *self-unlinked in B* if there exists a mapping $h: A \times [0,1] \to B$ such that

(a) $h|A \times \{0\} = f$,

(b) $h(A \times \{1\}) =$ a point,

(c) $h(A \times (0,1]) \subset B - f(A)$.

I.11. The term "*regular neighborhood*" is to be understood in the sense of Whitehead [17] with the additional requirement that a regular neighborhood is the closure of a topological neighborhood. We shall use the following well-known facts about regular neighborhoods of a subcomplex K of a subdivision α of the combinational manifold M:

(a) $\mathrm{st}(K, \alpha^2 M)$ is a regular neighborhood.

(b) There is a deformation retraction of any regular neighborhood of K onto K.

(c) Regular neighborhoods are combinatorial manifolds.

(d) Any two regular neighborhoods of K in M are pwl homeomorphic.

I.12. Let $f: J \to K$ be a pwl map of a 1-manifold into a 2-manifold. We shall call f *normal* if

(a) f is at most 2-to-1 and $S(f)$ is a finite set of points.

(b) $f(J)$ crosses itself at each point of $S(f)$.

I.13. Let $f:(D, \mathrm{Bd}\,D) \to (M^3, N^2)$ be a pwl map of a 2-manifold D with boundary into a 3-manifold M^3 with 2-submanifold N^2. We shall call f a *relative normal position* (*rnp*) *map* if

(a) $f(\mathrm{Bd}\,D) \cap f(D - \mathrm{Bd}\,D) = \varnothing$,

(b) f is at most 3-to-1,

(c) $f|\mathrm{Bd}\,D: \mathrm{Bd}\,D \to N$ is normal,

(d) $S(f)$ is a 1-dim polyhedron in M^3 consisting of (i) *double curves* (curves along which two sheets of $f(D)$ cross), (ii) *triple points* (points with arbitrarily small neighborhoods N such that $(N, N \cap f(D))$ is homeomorphic to (E^3, coordinate planes)), (iii) *branch points* (points with arbitrarily small closed neighborhoods N such that $(N, N \cap f(D))$ is homeomorphic to (C^3, cone from the origin over a singular curve on $\mathrm{Bd}\,C^3$)), and (iv) *pinched branch points* (points with arbitrarily small closed neighborhoods V such that $(V, V \cap f(D))$ is homeomorphic to (C^3, cone from origin over two paths on $\mathrm{Bd}\,C^3$)).

A (*crossing*) *pinch point* is a pinched branch point such that the paths on $\mathrm{Bd}\,C^3$ are disjoint arcs.

A *simple pinched branch* point is a pinched branch point such that the paths on $\mathrm{Bd}\,C^3$ are arcs with one common interior point.

Note that all pinched branch points must be crossing points of Bd D on N^2 because of (c).

I.14. If D_1 and D_2 are s-disks (s-spheres, loops) in M then D_1 is an *ε-alteration* of D_2 in M if $|D_1| \subset$ ε-neighborhood of $|D_2|$. D_1 is an *admissible ε-alteration* of D_2 if $|D_1| \subset (|D_2| +$ ε-neighborhood of $S(D_2))$, and $S(D_1) \subset$ ε-neighborhood of $S(D_2)$. D_1 is a *conservative ε-alteration* of D_2 in M if it is an admissible ε-alteration and $|D_2| -$ (ε-neighborhood of $S(D_2)) = |D_1| -$ (ε-neighborhood of $S(D_2))$.

(D_1, J_1) is a (conservative, admissible) ε-alteration of (D_2, J_2) in (M, N) if D_1 and J_1 are (conservative, admissable) ε-alterations of D_2 and J_2 in M and N, respectively. If $S(J_1) = \varnothing$, then N need not be mentioned.

I.15. If $f: K \to M$ and $f: L \to M$ are maps, then the *disjont sum of f and g* is a map $f \oplus g: K \oplus L \to M$ defined by

$$f \oplus g(x) = \begin{cases} f(x), & \text{if } x \in K, \\ g(x), & \text{if } x \in L, \end{cases}$$

where $K \oplus L$ denotes the abstract disjoint sum of K and L.

II. Normal position theorems

This part states two positioning theorems which will be used in other parts of this paper. The proofs of these theorems are contained in [9].

Theorem II. 2 ("Relative normal position theorem"). ***If*** *$f: (D, J) \to (M^3, N^2)$ is a pwl map of a pair into a pair, where D is a 2-manifold with boundary J and N^2 is a 2-submanifold of the 3-manifold M^3, and, in addition, $f(D - J) \cap f(J) = \varnothing, f|J$ is normal* (*see* I.12), *and ε is a positive number,* ***then*** *there exists a pwl map $g: (D, J) \to (M^3, N^2)$ such that*

(a) *g is obtained from f by a homotopy of (D, J) into (M^3, N^2) which moves each point less than ε and only moves points at all in an ε-neighborhood of the set of points in M^3 at which f fails to be in rnp* (*see* I.13),

(b) *g is a rnp map, and*

(c) $g|J = f|J$.

Theorem II.3. *Let N^2 be a 2-submanifold of the 3-manifold M^3 and h be a fixed-point-free pwl homeomorphism of (M^3, N^2) onto (M^3, N^2) such that hh equals the identity map. Let ε be a positive number.*

If *$f: (D, J) \to (M^3, N^2)$ is a pwl map of a 2-manifold D with boundary J into (M^3, N^2) such that $(f(D - J) + hf(D - J)) \cap (f(J) + hf(J)) = \varnothing$, and $f \oplus hf | J \oplus J$ is normal [$D \oplus D$ is the abstract disjoint union of two copies of D and $f \oplus hf$ is the mapping which is equal to f on one copy of D and hf on the other* (*see* I.15)], ***then*** *there exists a pwl map of $g: (D, J) \to (M^3, N^2)$ such that*

(a) *g is obtained from f by a homotopy of (D,J) into (M^3,N^2) which moves each point less than ε and only moves points at all in an ε-neighborhood of the set of points in M^3 at which $f \oplus hf$ fails to be in rnp.*

(b) *$g \oplus hg:(D_1,J_1) \oplus (D_2,J_2) \to (M^n,N^p)$ is a rnp map, and*

(c) *$g|J = f|J$.*

III. Loop Theorem

1. **Discussion of results.** C. D. Papakyriakopoulos first proved the Loop Theorem (for orientable 3-manifolds) in 1957 [11, Theorem 1]. Using a different proof, J. Stallings extended the Loop Theorem to nonorientable manifolds [15, §2]. In this part we shall give several extensions and refinements of those results. Many of the techniques used here will be those of [14, §3] and [15].

If L is a loop in the arc-wise connected space X, then L determines a conjugate class of members of $\pi_1(X)$. Following Stallings we shall let $L(X)$ denote this conjugate class. If Λ is a normal subgroup of $\pi_1(X)$, then $L(X) \subset \Lambda$ has a precise meaning.

THEOREM III. 1 (PAPAKYRIAKOPOULOS AND STALLINGS). *Let B be a component of the boundary of the 3-manifold M. Let L be a loop in B and Λ be a normal subgroup of $\pi_1(B)$.*

If *$L(B) \not\subset \Lambda$ and $L(M) = 1$,* ***then*** *there is a nonsingular disk D' such that $\mathrm{Bd}\,D' \subset B$ and $\mathrm{Bd}\,D'(B) \not\subset \Lambda$.*

This theorem is essentially the same as Stalling's statement of the Loop Theorem, but a proof is given here for completeness. Stallings gives an example [15, §6] which shows that III.1 is false if B is merely assumed to be a 2-submanifold (see I.5) of M. However, we do get the following extension:

THEOREM III.2. *Let B be a 2-submanifold of the 3-manifold M. Let L be a normal* (*see* I.12) *loop in B, and Λ be a normal subgroup of $\pi_1(B)$.*

If *$L(B) \not\subset \Lambda$ and L is self-unlinked* (*see* I.10) *in M,* ***then*** *there is an s-disk D'* (*see* I.9), *nonsingular except for crossing pinch points, such that $\mathrm{Bd}\,D' \subset B$ and $\mathrm{Bd}\,D'(B) \not\subset \Lambda$.*

III.2 would be more useful if an answer were known to the following.

QUESTION III.3. ***If*** *B is a 2-submanifold of the 3-manifold M and L is a loop in B which is self-unlinked in M,* ***then*** *is there a normal loop L' which is homotopic to L in B and self-unlinked in M?*

There appears to be a relationship between III.3 and Question IV.1.

The following theorem is used to reduce III. 2 to the case where L is the boundary of a polyhedral s-disk D^* such that $|\mathrm{int}\,D^*| \cap |=L| = \varnothing$.

THEOREM III. 4. ***If*** *L is a polyhedral loop which is self-unlinked in a 3-mani-*

fold M, ***then*** *L is the boundary of a polyhedral s-disk* D^* *in* M *such that* $|\operatorname{int} D^*| \cap |L| = \varnothing$.

As consequences of II.1 *and* II.2 *we may require that, if L is normal in a* 2-*sub-manifold* $N \subset M$, *then* (D^*, L) *is in rnp in* (M, N).

Changing the s-disk D^* to a nearby nonsingular disk D' with boundary on B is investigated in detail in

THEOREM III.5. *Let B be a polyhedral* 2-*submanifold of the* 3-*manifold M. Let L be a polyhedral loop which is normal in B*, Λ *be a normal subgroup of* $\pi_1(B)$, *and* ε *be a positive number.*

If $L(B) \not\subset \Lambda$ *and L bounds an s-disk* D^* *such that* (D^*, L) *is in rnp in* (M, B), *then there is an s-disk* D', *nonsingular except for crossing pinch points* (*see* I.13), *such that*

(1.1) $\operatorname{Bd} D'(B) \not\subset \Lambda$, *and*

(1.2) $(D', \operatorname{Bd} D')$ *is an admissible* ε-*alteration* (*see* I.14) *of* (D^*, L).

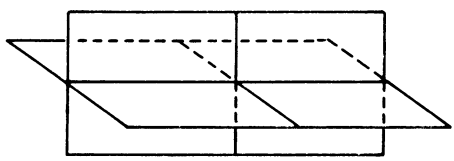
(1.3) *If* Λ *is the empty subgroup* (*that is to say,* "$X \not\subset \Lambda$" *is vacuously true for all X*), *then* $(D', \operatorname{Bd} D')$ *is a conservative* ε-*alteration of* (D^*, L). (*The condition on emptyness cannot be omitted.*)

(1.4) *If B is a component of* $\operatorname{Bd} M$, *then* D' *is nonsingular.*

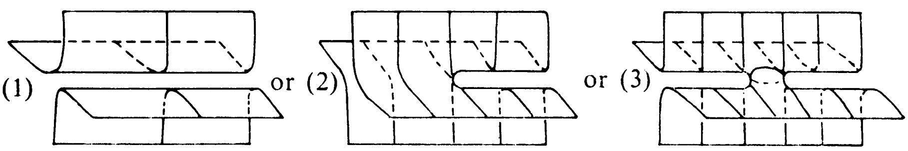
ADDENDUM TO III.5. *Let* $|D^*|$ *be a subcomplex of the subdivision* α *of M.*

If $(D', \operatorname{Bd} D')$ *is a conservative* ε-*alteration of* (D^*, L), *then* $|D'|$ *can be described relative to* $|D^*|$ *as follows*:

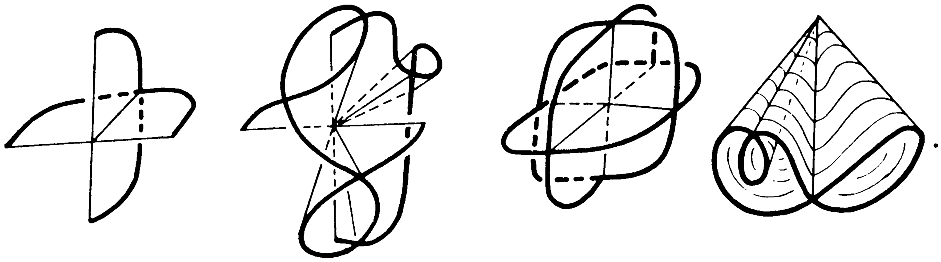
(a) *Along the double curves of* $|D^*|$, *between the triple, branch, and pinched branch points,* $|D^*|$ *looks like*



and $|D'|$ *looks like*

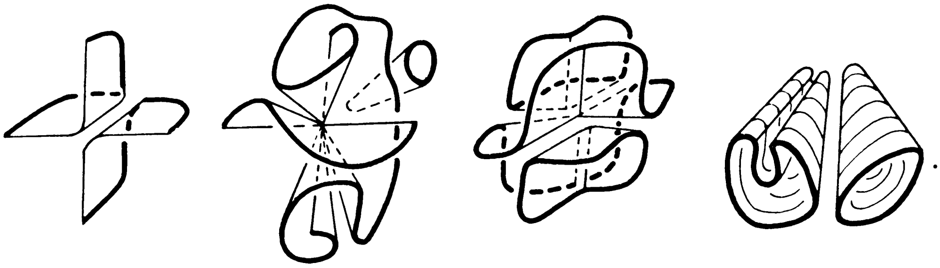


(b) *Near a triple, branch, or pinched branch point, p,* $|D^*|$ *looks like the cone over one or two singular curves on the boundary of a small ball C about p; for example,*



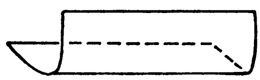
pinched branch points triple point branch point

The part of $|D'|$ *along the double curves determines* $|D'|$ *on the boundary of* C. $|D'| \cap \operatorname{Bd} C$ *is a finite collection of disjoint scc's and at most two arcs, and* $|D'| \cap C$ *is a finite collection of disks which "cap off" each scc and arc.* (*A disk that "caps off" an arc contains the arc on its boundaries and the remaining portion of the boundary is on* $B \cap C$) *The disks of* $|D'| \cap C$ *are disjoint except that the two disks which "cap off" arcs may have a common boundary point* (*a crossing pinch point*). *For example*



crossing pinch point of D'

Furthermore, if $(D', \operatorname{Bd} D')$ *is an admissible* (*but not conservative*) ε*-alteration of* (D^*, L), *then* $|D'|$ *can be described exactly as above except that, in* (a), $|D'|$ *may in addition look like*



or be empty along a double curve of $|D^*|$; *and, in* (b) $|D'| \cap C$ *might be empty.*

Theorems III.4 and III.5 have several applications in later parts of this paper. The addendum to III.5 will be used only in the proof of Theorem V.6 of [8].

The theorems of this part will be proved in the logical, not numerical, order: III.4, III.5, III.2, III.1.

2. **Proof of III. 4.** $|L|$ is a finite 1-dim graph and we can consider it as a subcomplex of some subdivsion α of M. Since L is self-unlinked we may assume that L is the boundary of an s-disk D, such that $|L| \cap |\operatorname{int} D| = \varnothing$. We then

suppose that each of $\operatorname{int}\Delta$ (Δ is the standard disk) and $\alpha M - |L|$ are so subdivided that given a positive number δ, there are only finitely many simplexes with diameters larger than δ. Then applying, to $D|\operatorname{int}\Delta:\operatorname{int}\Delta \to \alpha M - |L|$, a simplicial approximation theorem for infinite complexes (see [13, p. 115]), we may assume that $(D|\operatorname{Bd}\Delta) = L$ is simplicial and that $D|\operatorname{int}\Delta$ takes simplexes linearly into simplexes (of $\alpha M - |L|$).

Let Δ' be a disk in $\operatorname{int}\Delta$ so large that there exists a homeomorphism $h:\operatorname{Bd}\Delta' \to \operatorname{Bd}\Delta$ such that

(2.1) h takes simplexes of $\operatorname{Bd}\Delta'$ linearly into simplexes of $\operatorname{Bd}\Delta$,

(2.2) for every 1-simplex σ in $\operatorname{Bd}\Delta$, there is a 1-simplex $\sigma' \subset h^{-1}(\sigma) \subset \operatorname{Bd}\Delta'$ such that $D(\sigma') \subset \operatorname{ost}(\operatorname{int} D(\sigma), \alpha^2 M)$, and

(2.3) if σ_1 and σ_2 are 1-simplexes of $\operatorname{Bd}\Delta$ and $v = \sigma_1 \cap \sigma_2$, then

$$D\,(\text{``small'' component of } \operatorname{Bd}\Delta' - (\sigma_1' + \sigma_2'))$$
$$\subset \operatorname{ost}(\operatorname{int} D(\sigma_1 + \sigma_2), \alpha^2 M) \subset \operatorname{ost}[\operatorname{ost}(D(v), |L|), \alpha^2 M] \equiv O(v).$$

[That such a homeomorphism exists follows from the uniform continuity of D and the fact that the diameters of simplexes of $\operatorname{int}\Delta$ form a null sequence.]

For each 1-simplex $\sigma \in \operatorname{Bd}\Delta$ let $\Delta(\sigma)$ be a disk in $\Delta - \operatorname{int}\Delta'$ such that

(2.3a) $\operatorname{Bd}\Delta(\sigma) \supset \sigma + \sigma'$ (see (2.2)),

(2.4) $\Delta(\sigma_1) \cap \Delta(\sigma_2) = \sigma_1 \cap \sigma_2$ for each pair of 1-simplexes in $\operatorname{Bd}\Delta$, and

(2.5) $\Delta(\sigma) \cap \Delta' = \operatorname{Bd}\Delta(\sigma) \cap \operatorname{Bd}\Delta' = \sigma'$.

For each vertex $v \in \operatorname{Bd}\Delta$, let $\Delta(v)$ be the unique closure of the component of $\Delta - (\Delta' + \sum\Delta(\sigma))$ that contains v. (See Figure 1.)

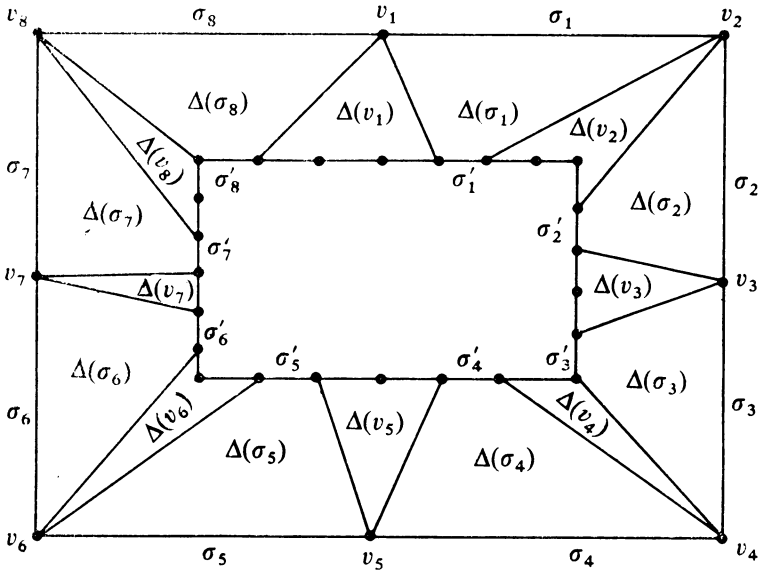


FIGURE 1

There is a homeomorphism g_i such that the pair $[\mathrm{st}(\mathrm{int}\, D(\sigma_i),\alpha^2 M), D(\sigma_i) + D(\sigma_i')]$ is pwl homeomorphic under g_i to $[C^3,(C^3 \cap x\,\text{axis})$ + (the straight line segment from $(\frac{1}{2},\frac{1}{2},0)$ to $(-\frac{1}{2},\frac{1}{2},0))]$. (See (2.2) and Figure 2.)

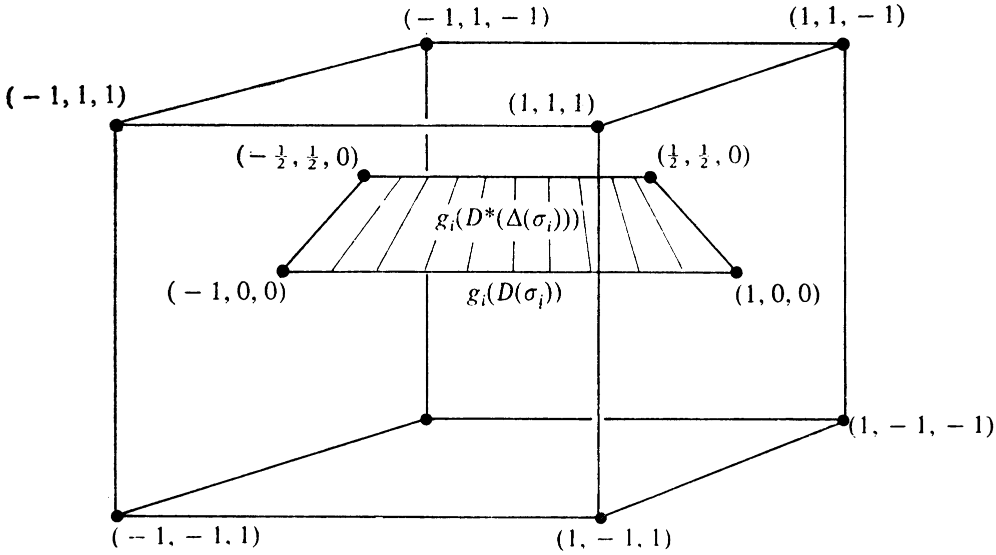


FIGURE 2

There is a homeomorphism f_i such that the triple

$$[O(v_i), O(v_i) \cap |L| = \mathrm{st}(O(v_i), |L|), v_i]$$

is pwl homeomorphic under f_i to $[C^3$, finite number of radii of C^3, origin]. (See (2.3) and Figure 3.)

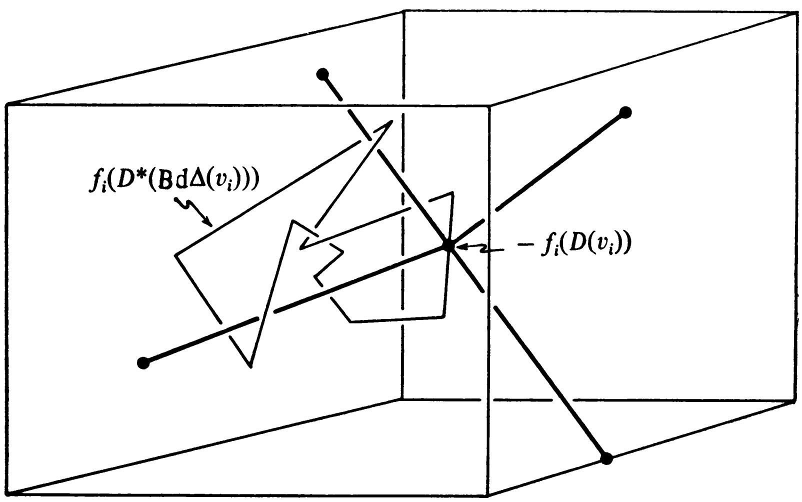


FIGURE 3

Define $D^*: \Delta \to M$ by stipulating the following:

(2.6) $D^* | \Delta' + \mathrm{Bd}\,\Delta = D | \Delta' + \mathrm{Bd}\,\Delta$.

(2.7) For each i, $D^* | \Delta(\sigma_i)$ takes $\Delta(\sigma_i)$ pwl onto g_i^{-1} [join of $g_i(D(\sigma_i'))$ and $g_i(D(\sigma_i))$] (see Figure 2) so that it matches correctly on $\sigma_i + \sigma'_i$.

(2.8) For each i, $D^* | \Delta(v_i)$ takes $\Delta(v_i)$ pwl onto f_i^{-1} [cone from $f_i(D(v_i))$ over $f_i(D^*(\mathrm{Bd}\,\Delta(v_i)))$] so that it matches correctly on $\mathrm{Bd}\,\Delta(v_i)$. (See Figure 3.)

We can accomplish (2.8), since (2.3), (2.6) and (2.7) require that $D^*(\mathrm{Bd}\,\Delta(v_i)) \subset O(v_i)$.

D^* is the s-disk required by III.4.

3. **Proof of III. 5.** First of all, note that (1.4) of III. 5 follows from the rest since a point p of $\mathrm{Bd}\,D'$ cannot be a crossing point and a pinch point without having D' intersect both "sides" of B near p.

Let $\mathfrak{B}$ be the collection of all quintuples $(B, M, \Lambda, D^*, \varepsilon)$ which satisfy the hypothesis of III.5, where $|D^*|$ is considered to have a fixed triangulation as a subcomplex of αM, for some subdivision α of M. Let $\mathfrak{B}^*$ be the subcollection of $\mathfrak{B}$ which consists of all those quintuples which do not satisfy the conclusion of III.5, including the Addendum. Let $\mathfrak{B}_1$ be the subcollection of $\mathfrak{B}$ consisting of all quintuples $(B, M, \Lambda, D^*, \varepsilon)$ for which

(a) there is a 3-submanifold P of M such that $(\mathrm{int}\,D^*, \mathrm{Bd}\,D^*) \subset (\mathrm{int}\,P, \mathrm{Bd}\,P)$ [P need have no other relationship to B], and

(b) each pinched branch point of D^* is simple.

Let $\mathfrak{B}_1^* = \mathfrak{B}_1 \cap \mathfrak{B}^*$. We wish to show that $\mathfrak{B}^*$ is empty. The first, and largest, step toward this end is to show (Proposition III.15) that $\mathfrak{B}_1^* = \varnothing$.

Let D_i^* be the fourth element of $(B_i, M_i, \Lambda_i, D_i^*, \varepsilon_i) \in \mathfrak{B}$ and suppose that $|D_i^*|$ has a fixed triangulation as a subcomplex of $\alpha_i M_i$, for $i = 1, 2$. We shall say that $D_1^* < D_2^*$ if

(3.1) there is a simplicial map f of $|D_1^*|$ onto $|D_2^*|$ such that $fD_1 = D_2$, and

(3.2) $f(S(D_1^*))$ is properly contained in $S(D_2^*)$.

Since D_i^* is a rnp map, it is at most 3-to-1, and thus $f | S(D_1^*)$ is 1-to-1 into $S(D_2^*)$. In addition, $S(D_i^*)$ is a subcomplex of $|D_i^*|$; therefore, $S(D_1^*)$ has fewer simplexes than $S(D_2^*)$. It should now be clear that

(3.3) there is a member $(B, M, \Lambda, D^*, \varepsilon)$ of $\mathfrak{B}_1^*$ such that if $D_1^* < D^*$ then D_1^* is not the fourth element of any member of $\mathfrak{B}_1^*$.

We shall show that the existence of such a member of $\mathfrak{B}_1^*$ leads to a contradiction by considering two cases.

Let δ be a subdivision of M such that $|D^*|$ is a subcomplex of δM and $V = \mathrm{st}(|D^*|, \alpha M)$ is contained in the ε-neighborhood of $|D^*|$, where $\alpha = \delta^2_{|D^*|}$ (see I.4).

3a. ***Case* 1.** *V has a 2-sheeted covering.*

Let $f: M_1 \to V$ be a 2-sheeted covering of V and let D_1^* be an s-disk in V_1 such that $fD_1^* = D^*$. f^{-1} induces a triangulation τ of M_1 such that f is simplicial from

τM_1 onto $V \subset \alpha M$. We assume that M_1 has this triangulation, and that the metrics d and d_1 for αM and τM_1, respectively, are the barycentric metrics so that, if $x, y \in \tau M_1$ and $d_1(x,y) < 1$, then $d(f(x), f(y)) = d_1(x,y)$. Without loss of generality we assume that $\varepsilon < 1$.

Define $B_1 = f^{-1}(V \cap B)$, and $\Lambda_1 = (f|B_1)_*^{-1}(\Lambda)$, where $(f|B_1)_*$ is the natural induced homomorphism of $\pi_1(B_1)$ into $\pi_1(B)$. Let ε_1 be so small and m so large that, for every vertex v in $\tau^m M_1$,

$$\varepsilon_1\text{-neighborhood of } v \subset \operatorname{st}(v, \tau^{m+2} M_1) \subset \varepsilon\text{-neighborhood of } v.$$

PROPOSITION III.6. *$(B_1, M_1, \Lambda_1, D_1^*, \varepsilon_1) \in \mathfrak{B}_1$ and Λ_1, is normal (or empty) if Λ is normal (or empty).*

The straightforward verification of this proposition is left to the sufficiently interested reader.

PROPOSITION III.7. $D_1^* < D^*$.

We prove this by adopting an argument of Stallings' [15, p. 17] to show that $f(S(D_1^*))$ not $= S(D^*)$. For if $f(S(D_1^*)) = S(D^*)$, then $f|\,|D_1^*|$ is 1-to-1 onto $|D^*|$; and thus $f|\,|D_1^*|$ is a homeomorphism. The following is a commutative diagram, where the horizontal maps are induced by inclusions:

$$\begin{array}{ccc} \pi_1(|D_1^*|) & \xrightarrow[j]{} & \pi_1(M_1) \\ {\scriptstyle (f|\,|D_1^*|)_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \pi_1(|D^*)| & \xrightarrow{k} & \pi_1(V) \end{array} .$$

Since $|D^*|$ is a deformation retract of V (see I.11), and $f|\,|D_1^*|$ is a homeomorphism, k and $(f|\,|D_1^*|)_*$ are isomorphisms onto. However, the image of f_* is a subgroup of index two in $\pi_1(V)$ since $f: M_1 \to V$ is a double covering. This is a contradiction.

Therefore $D_1^* < D^*$ and by our assumption on D^*, $(B_1, M_1, \Lambda_1, D_1^*, \varepsilon_1) \notin \mathfrak{B}^*$. Therefore, there is an s-disk, D_1', nonsingular except for crossing pinch points which satisfies (1.1)–(1.3) and the Addendum of III.5.

Let h be the nontrivial covering translation of (M_1, B_1) such that $hh =$ identity. Since triple points are the only points of D^* with more than two preimages, and since D^* has only simple pinched branch points,

$$f[(h(|D_1^*|) \cap S(D_1^*)) + (|D_1^*| \cap h(S(D_1^*)))] \subset \text{set of triple points of } D^* \subset |\operatorname{int} D^*|,$$

and $h(|D_1'|) \cap |D_1'|$ differs from $h(|D_1^*|) \cap |D_1^*|$ only near $(h(|D_1^*|) \cap S(D_1^*)) + (|D_1^*| \cap h(S(D_1^*)))$. Therefore by using a smaller ε, if necessary, we may assume that

$$(|\operatorname{int} D_1'| + h|\operatorname{int} D_1'|) \cap (|\operatorname{Bd} D_1'| + h|\operatorname{Bd} D_1'|) = \varnothing$$

and thus, by II.3, we may also assume that $(D_1' \oplus hD_1', \operatorname{Bd} D_1' \oplus \operatorname{Bd} hD_1')$ is in rnp in (M_1, B_1).

We see that $S(fD_1') = f(S(D_1' + hD_1')) \subset S(D^*) + \varepsilon$-neighborhood of the triple points of D^*. Thus fD_1' is a (singular) s-disk which satisfies (1.1)–(1.5) of III.5. Also, f is a local homeomorphism and at most 2-to-1 on D_1'.

By standard arguments $S(fD_1')$ can be seen to be a finite union of pinch points and double curves which are disjoint simple closed curves (scc's) in $\operatorname{int} fD_1'$ or spanning arcs of fD_1'. For each scc J in $S(fD_1')$, $(fD_1')^{-1}(J)$ is either one scc or a pair of disjoint scc's in $\operatorname{int}\Delta$ (the standard disk). For each spanning arc A in $S(fD_1')$, $(fD_1')^{-1}(A)$ is a pair of disjoint spanning arcs of Δ.

Standard arguments in [15, p. 14–15] (see also [12, #14]) show how to get an s-disk D'' such that

(3.4)
- (a) $\operatorname{Bd} D'' = \operatorname{Bd} fD_1'$,
- (b) D'' is a conservative $\varepsilon/3$-alteration of fD_1', and
- (c) $S(D'')$ consists of a finite collection of disjoint spanning arcs and pinch points,
- (d) $|D''|$ is gotten from $|fD_1'|$ by cuts along double curves of the type pictured in the Addendum (a)(1); and thus
- (e) $(B, M, \Lambda, D'', \varepsilon)$ satisfies (1.1)–(1.3).

Suppose $S(D'')$ has a spanning arc A and let $D''^{-1}(A) = A_1 + A_2 \subset \Delta$. Let $D'' = EF$ where F is an s-disk such that $S(F) = F(A_1 + A_2)$ and, for $a_1 \in A_1$ and $a_2 \in A_2$, $F(a_1) = F(a_2)$ if and only if $D''(a_1) = D''(a_2)$. There are two cases depending on whether $|F|$ is like Figure 4a or Figure 4b. We shall assume that $|F|$ looks like Figure 4a with the idea that the reader, if he wishes, could supply the changes necessary in the proof for the case that $|F|$ looks like Figure 4b.

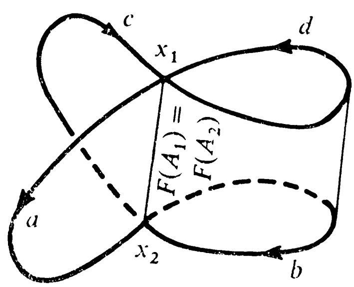


FIGURE 4a

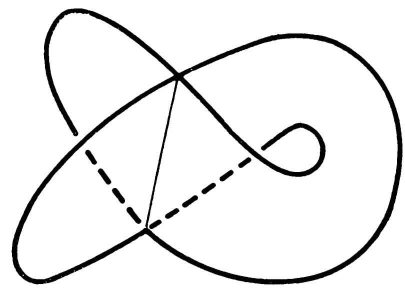


FIGURE 4b

Consider the s-disks F_1 and F_2, illustrated in Figure 5. The s-disks are to be so arranged that

(a) $(EF_1, \mathrm{Bd}\, EF_1)$ is an admissable $\varepsilon/3$-alteration of $(D'' = EF, \mathrm{Bd}\, D'')$,

(b) $(EF_2, \mathrm{Bd}\, EF_2)$ is a conservative $\varepsilon/3$-alteration of $(D'', \mathrm{Bd}\, D'')$, and

(c) for each i, EF_i satisfies (c) and (d) of (3.4) and (1.1)–(1.3) of III.5, except possibly the condition that $\mathrm{Bd}\, EF_i(B) \notin \Lambda$.

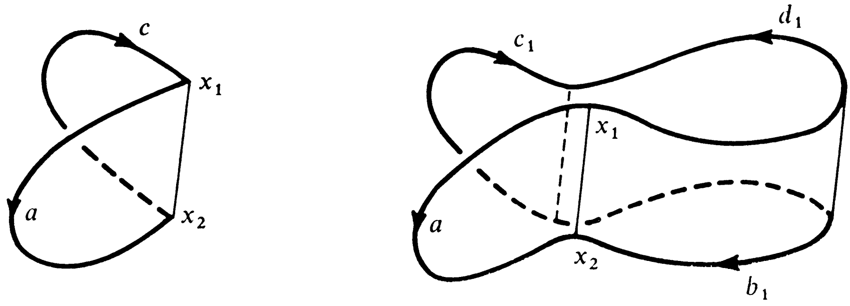


FIGURE 5

It can be seen that, for each i, $S(EF_i)$ has fewer spanning arcs than $S(D')$. We wish to conclude that, for some i, $\mathrm{Bd}\, EF_i(B) \notin \Lambda$. If Λ is the empty subgroup, this is immediate. (See (1.3) of III.5.)

To show that we have the desired conclusion if Λ is nonempty, consider the paths labelled in Figures 4 and 5. Select $E(x_1)$ as the base point for $\pi_1(B)$ and let λ be a path from $E(x_2)$ to $E(x_1)$ in B. Let a' = conjugate class of $\pi_1(B)$ represented by the loop that follows $E(a)$ and then λ; b' = conjugate class of $\pi_1(B)$, represented by the loop that follows λ^{-1}, $E(b)$, and then λ; c' = conjugate class of $\pi_1(B)$ represented by the loop that follows λ and then $E(c)$; and d' = conjugate class of $\pi_1(B)$ represented by the loop $E(d)$. It is seen that $\mathrm{Bd}\, D''(B) = a'b'c'd'$, $\mathrm{Bd}\, EF_1(B) = a'c'$, and $\mathrm{Bd}\, EF_2(B) = a'b'^{-1}c'd'^{-1}$. Therefore, if $\mathrm{Bd}\, EF_1(B)$ and $\mathrm{Bd}\, EF_2(B)$ belong to Λ,

$$\mathrm{Bd}\, D''(B) = a'b'c'd' = a'c' \cdot c'^{-1}b'c'[(a'b'^{-1}c'd'^{-1})^{-1} \cdot a'c']c'^{-1}b'^{-1}c' \in \Lambda,$$

since Λ is normal. This is a contradiction.

Thus we have obtained an s-disk satisfying (c), (d), and (e) of (3.4) and with fewer spanning-arc-double-curves than D''. Continuing in this fashion, we can obtain an s-disk D' which satisfies the conclusion of III.5 except that it might have noncrossing pinch point singularities. But Proposition III.8 (to follow) applies, and thus $(B, M, \Lambda, D^*, \varepsilon) \notin \mathfrak{B}_1$ in contradiction to (3.3).

We finish §3a with a proof of

PROPOSITION III.8. *Let D be a polyhedral s-disk and B be a 2-submanifold of the 3-manifold M, such that $(D, \mathrm{Bd}\, D) \subset (M, B)$. Let ε be a positive number.*

***If** D has only pinch point singularities, **then** there is an s-disk D', with only crossing pinch point singularities, such that $(D', \mathrm{Bd}\, D')$ is a conservative ε-alteration of $(D, \mathrm{Bd}\, D)$.*

Let α be a subdivision of M so that $|D|$ is a subcomplex of αM and for every $v \in S(D)$, $\mathrm{st}(v,\alpha^2 M)$ is contained in the ε-neighborhood of v.

If v is a noncrossing pinch point of $S(D)$, then the triple

$$(\mathrm{st}(v,\alpha^2 M),\ \mathrm{st}(v,\alpha^2 M) \cap B,\ \mathrm{Bd}(\mathrm{st}(v,\alpha^2 M)) \cap |D|)$$

is pwl homeomorphic under g to ($C^3, C^3 \cap xy$-plane, two disjoint arcs (A_1, A_2) on $\mathrm{Bd}\, C^3$ with endpoints on xy-plane). Since v is a noncrossing point $\mathrm{Bd}\, A_1$ does not separate $\mathrm{Bd}\, A_2$ on (xy-plane) $\cap \mathrm{Bd}\, C^3$. Thus there is a pwl isotopy $H: \mathrm{Bd}\, C^3 \times [0,1] \to \mathrm{Bd}\, C^3$, such that (a) $H((A_1 + A_2) \times \{1\}) \subset \mathrm{Bd}\, C^3 \cap xy$-plane, (b) $H | \mathrm{Bd}\, C^3 \times \{0\} = $ identity, and (c) $H(\mathrm{Bd}\, A_1 + \mathrm{Bd}\, A_2 \times [0,1]) \subset xy$-plane.

For $t > 0$, let m_t be the natural linear map of $\mathrm{Bd}\, C^3$ onto the boundary of the cube C_t^3 which has as vertices the eight points of E^3 each of whose coordinates are $\pm 1/t$.

Now, define $F: A_1 + A_2 \times [0,1] \to C^3$ by $F(a,t) = m_{1-t/2}(H(a,t))$. $F(A_1 + A_2 \times [0,1])$ is two disjoint disks E_1, E_2 in C^3 such that $(\mathrm{Bd}\, E_1 + \mathrm{Bd}\, E_2) \cap \mathrm{int}\, C^3 \subset xy$-plane and $\mathrm{Bd}\, E_i \cap \mathrm{Bd}\, C^3 = A_i$ for $i = 1,2$.

The construction of an s-disk D' to satisfy III.8 is now easy.

3b. ***Case* 2.** *V has no 2-sheeted covering.*

The property of having or not having a 2-sheeted covering is an invariant of the fundamental group [13, p. 188].

Let β be a subdivision of P such that for every subcomplex $K \subset |D^*|$, $\mathrm{st}(K, \beta^2 P) \subset \varepsilon$-neighborhood K. Simple, but tedious, calculations show that

$$V' \equiv \mathrm{st}[|D^*| + \mathrm{st}(S(D^*), \beta^4 P), \beta^6 B] \subset \mathrm{st}(|D^*|, \beta^2 P)$$

and that

$$\mathrm{st}(S(D^*), \beta^4 P) = \mathrm{st}[\mathrm{ost}(S(D^*), \beta^4 P \cap |D^*|), \beta^4 P]$$

and that therefore

$$|D^*| \text{ is a deformation retract of } V'.$$

(The details of the definitions of V' will only be used in III.13.)

Therefore $\pi_1(V') = \pi_1(|D^*|) = \pi_1(V)$ and V' has no 2-sheeted covering. Also in [14, Proof of Lemma (3.1)] it is shown

PROPOSITION III.9. *Each component of* $\mathrm{Bd}\, V'$ *is a 2-sphere.*

If $H_1(V')$ has an element of infinite or even order, then one of the generators of $H_1(V')$ has infinite or even order and thus there is an epimorphism $g: H_1(V') \to Z_2$. If h is the Hurewicz epimorphism of $\pi_1(V')$ onto $H_1(V')$ then the kernel of gh is a subgroup of $\pi_1(V')$ of index 2 and thus [13, p. 188] V' would have a 2-sheeted covering. This contradiction proves

PROPOSITION III.10. *Each element of $H_1(V')$ has finite odd order.*

If J is a scc(arc) in V' then we shall say that J *is normal wrt* D^* if (a) J is polygonal, (b) $|J| \cap (S(D^*) + |\mathrm{Bd}\, D^*|) = \varnothing$, and (c) J crosses D^* at each point of $|D^*| \cap |J|$. If the scc J is normal wrt D^*, then J represents a 1-cycle in $H_1(V')$ of odd order (III.10). Therefore J bounds a 2-chain, $\mathscr{C}$, with mod 2 coefficient in V'. We may suppose that $\mathscr{C}$ is a simplicial chain such that $|\mathscr{C}| \cap |\mathrm{Bd}\, D^*| = \varnothing$. $|\mathscr{C}|$ is a 2-complex each of whose 1-simplexes is the face of an even member of 2-simplexes if and only if it does not lie in $|J|$. By employing a slight push that does not change the cardinality of $|J| \cap |D^*|$, we may assume that the vertices of $|\mathscr{C}|$ miss $|D^*|$. Then $D^{*-1}(|\mathscr{C}| \cap |D^*|)$ is a finite 1-dimensional graph whose only vertices of odd order are the points of $D^{*-1}(|J| \cap |D^*|)$. But the sum of the orders of all the vertices of a finite 1-dimensional graph is equal to twice the numbers of edges; thus the cardinality of $|J| \cap |D^*|$ (= cardinality of $D^{*-1}(|J| \cap |D^*|)$) is even and we have

PROPOSITION III.11. *Every scc J in V' which is normal wrt D^* intersects $|D^*|$ an even number of times.*

Let K be a subcomplex of the subdivision γ of the 3-manifold P, and look at $U = \mathrm{ost}(K, \gamma^1 P)$ and $\mathrm{cl}(U) = \mathrm{st}(K, \gamma^1 P)$. Every 3-simplex of $\mathrm{cl}(U)$ is the join of a face in K and a face in $\mathrm{cl}(U) - U$. For $p \in U$ let $R_1(p) =$ the unique point of K such that p and $R_1(p)$ are on a line segment from K to $\mathrm{cl}(U) - U$ determined by the join structure of a simplex of $\mathrm{cl}(U)$.

It can be checked that R_t $(0 \leqq t \leqq 1)$: $U \to U$ is a pwl pseudo-isotopy if it is defined by setting $R_t(p)$ equal to the point t of the way from p to $R_1(p)$ along the line segment joining p to $R_1(p)$.

A straightforward geometric argument shows that for every $p \in \mathrm{st}(K, \gamma^2 P)$, $R_t(p) \in \mathrm{ost}(K, \gamma^2 P)$, for $0 < t \leqq 1$.

Therefore it can be seen that $H: [\mathrm{st}(K, \gamma^2 P) - \mathrm{ost}(K, \gamma^2 P)] \times (0,1) \to [\mathrm{st}(K, \gamma^2 P) - K]$ is a homeomorphism if H is defined by $H(p,t) = R_t(p)$.

Thus

PROPOSITION III.12. *For any subcomplex K of a sub-division γ of a 3-manifold P, $\mathrm{ost}(K, \gamma^2 P) - K$ is homeomorphic to $[\mathrm{st}(K, \gamma^2 P) - \mathrm{ost}(K, \gamma^2 P)] \times (0,1)$.*

We can now prove

PROPOSITION III.13. *Let* $V_0 = \mathrm{ost}[(|D^*| + S_0) \cap \mathrm{Bd}\, P, \beta^6\, \mathrm{Bd}\, P]$ *and* $S_0 = \mathrm{st}(S(D^*), \beta^4 P)$.

***If** Q is the closure of a component of $V' - (|D^*| + S_0)$, **then** there is a homeomorphism $h: X \times [0,1] \to Q$ such that*

(a) *X is either a disk or 2-sphere,*

(b) $h(X \times 1) \subset \mathrm{Bd}\, V' - V_0$,

(c) $h(X \times 0) \subset \mathrm{Bd}(|D^*| + S_0)$, *and*

(d) $h(X \times (0,1]) \subset V' - (|D^*| + S_0)$.

We first show that Q is a manifold with $Q \cap (|D^*| + S_0) \subset \mathrm{Bd}\, Q$. III.11 shows that $Q \cap |D^*| - S_0 \subset \mathrm{Bd}\, Q$. Also $Q \cap S_0 \subset \mathrm{Bd}\, S_0$ therefore $Q \cap S_0 \subset \mathrm{Bd}\, Q$. Q is clearly a 3-manifold except possibly at points of $Q \cap |D^*| \cap S_0$. But for $p \in Q \cap |D^*| \cap S_0$, $\mathrm{st}(p, \beta^6 P) \cap S_0 = \mathrm{st}(p, \beta^6 P \cap S_0) =$ combinatorial 3-ball with p on its boundary, thus $(\mathrm{st}(p, \beta^6 P) - \mathrm{int}\, S_0)$ is a combinatorial 3-ball and $(\mathrm{st}(p, \beta^6 P) - \mathrm{int}\, S_0) \cap |D^*| + S_0)$ is three disks E_1, E_2, E_3 which intersect along a common arc $A \subset \mathrm{Bd}\,\mathrm{st}(S(D^*), |D^*|)$ such that $p \in A$ and $E_1 + E_2 + \mathrm{Bd}\, E_3 \subset \mathrm{Bd}(\mathrm{st}(p, \beta^6 P) - S_0)$. Since Q intersects only one side of $|D^*| - S_0$ (see III.11), it is seen that $\mathrm{st}(p, \beta^6 Q)$ is a 3-ball with $\mathrm{st}[p, (|D^*| + S_0) \cap \beta^6 Q]$ a disk on the boundary of $\mathrm{st}(p, \beta^2 Q)$. Thus Q is a manifold and $(\mathrm{Bd}\, V' + |D^*| + S_0) \cap Q \subset \mathrm{Bd} Q$.

Since $\mathrm{Bd}\, V' - V_0 = \mathrm{st}(|D^*| + S_0, \beta^6 P) - \mathrm{ost}(|D^*| + S_0, \beta^6 P)$, by III.12 and III.9, $\mathrm{int}\, Q$ is homeomorphic to (2-sphere $\times (0,1)$) or $Q \cap (\mathrm{Bd}\, V' - V_0)$ is a disk and $[\mathrm{int}\, Q + (Q \cap V_0 - (|D^*| + S_0))]$ is homeomorphic to (disk $\times (0,1)$). By a 2-dimensional version of III.12, $V_0 - (|D^*| + S_0)$ is homeomorphic to (1-sphere $\times (0,1)$) and thus $\mathrm{int}\, Q$ is either a (2-sphere $\times (0,1)$) or (the interior of a disk $\times (0,1)$). III.13 now follows from a theorem of Edwards [7, Theorem 3] to the effect that two compact 3-manifolds are homeomorphic if their interiors are.

Since $\mathrm{Bd}\, D^*$ is normal in B, $|\mathrm{Bd}\, D^*|$ is a 1-dim finite graph with no free vertices. Let $J_1, \cdots, J_k$ be the loops of $|\mathrm{Bd}\, D^*|$ which are innermost on $\mathrm{Bd}\, V'$, i.e., each J_i is the boundary of a disk whose interior is nonsingular and contained in $\mathrm{Bd}\, V' - |\mathrm{Bd}\, D^*|$. The set of all loops L in $|\mathrm{Bd}\, D^*|$ for which $L(B) \in \Lambda$ represents a normal subgroup Λ' of $\pi_1(|\mathrm{Bd}\, D^*|)$. If $J_1(|\mathrm{Bd}\, D^*|), \cdots, J_k(|\mathrm{Bd}\, D^*|)$ each belong to Λ', then it is easy to check that $\Lambda' = \pi_1(|\mathrm{Bd}\, D^*|)$ using the standard representations of the fundamental group of a finite complex. (E.g., see [13, Siebentes Kapitel].) Thus we have

PROPOSITION III.14. *There is a loop* $J \subset |\mathrm{Bd}\, D^*|$ *such that* $J(B) \notin \Lambda$ *and* J *bounds an s-disk E whose interior is nonsingular and contained in* $\mathrm{Bd}\, V' - |\mathrm{Bd}\, D^*|$.

Let Q be the closure of the component of $P - (|D^*| + S_0)$ that intersects E. $E \cap \mathrm{Bd}\, Q$ is a nonsingular disk whose interior is in $P - (|D^*| + S_0)$. By III.13, $\mathrm{Bd}(E \cap \mathrm{Bd}\, Q)$ bounds an s-disk D'' (nonsingular) in $|D^*| + S_0$. Also $|\mathrm{Bd}\, D''| - S_0 = |J| - S_0$, thus adding the appropriate subdisks of $S_0 \cap \mathrm{Bd}\, P$ to D'' we obtain an s-disk D' which is nonsingular except on $J = \mathrm{Bd}\, D'$. It can be seen that $|D'|$ satisfies the Addendum. Therefore, if Λ is nonempty, D' satisfies the conclusion to III.5 except that it might have noncrossing pinch points. Thus III.8 allows us to conclude that $(B, M, \Lambda, D^*, \varepsilon) \notin \mathfrak{B}_1^*$ which directly contradicts one of our assumptions.

If Λ is empty, we may ignore it. Let $E_1, \cdots, E_k$ be a maximal collection of components of $\mathrm{Bd}\, V' - V_0$ such that each can be reached from each other by an arc in V' which is normal wrt D^* and which crosses $|D^*|$ an even number of times. By III.11 there is such a collection, and using III.13 we conclude that if

$E'_i = h(X \times 0)$ when $E_i = h(X \times 1)$, then $E'_1, \cdots, E'_k$ is a collection of disjoint non-singular s-disks and s-spheres such that $|D^*| - S(D^*) \subset \sum_{i=1}^{k}(E'_i)$.

Let $E'_1, \cdots, E'_r$ be the disks of $E'_1, \cdots, E'_k$. Then each E_i, $i = 1, \cdots, r$, is contained in a fixed component C of $\mathrm{Bd}\, V'$ and are separated from each other on C by $|\mathrm{Bd}\, D^*|$.

Let H be a 1-dim graph with one vertex for each of the s-disks $E'_1, \cdots, E'_r$. Let $v_1, \cdots, v_r$ be the vertices of H that correspond to $E'_1, \cdots, E'_r$. Let H contain an edge e_{ij} joining v_i to v_j, $i < j$, if and only if, $\mathrm{Bd}\, D'_j$ can be connected to $\mathrm{Bd}\, E'_j$ by an arc λ_{ij} in $S_0 \cap \mathrm{Bd}\, P$. Since each point of $\mathrm{Bd}\, P \cap S(|\mathrm{Bd}\, D^*|)$ is a crossing point of order 2, the λ_{ij}'s must be disjoint.

We can use III.11 to show that H is connected and thus has a maximal connected tree H'. For each $e_{ij} \in H'$, connect E'_i to E'_j by a small pinched strip within ε of $S(\mathrm{Bd}\, D^*)$ as shown in Figure 6. (The strip should be polygonal and in $\mathrm{int}\, P + |\mathrm{Bd}\, D^*|$.) If p is the point of $S(\mathrm{Bd}\, D^*)$ involved; then, since p is a simple pinched branch point, along the double curve of $S(D^*)$ leading to p the new disk looks like the illustration (a)(2) of the Addendum.

Expand $\sum_{i=1}^{r}(E'_1)$ out to $\mathrm{Bd}\, D^*$ by adding the small disks of $S_0 \cap \mathrm{Bd}\, P$ with boundaries on $\sum_{i=1}^{r}(\mathrm{Bd}\, E'_i) + \mathrm{Bd}\, D^*$, except near those λ_{ij} for which $e_{ij} \in H'$. (See Figure 6.)

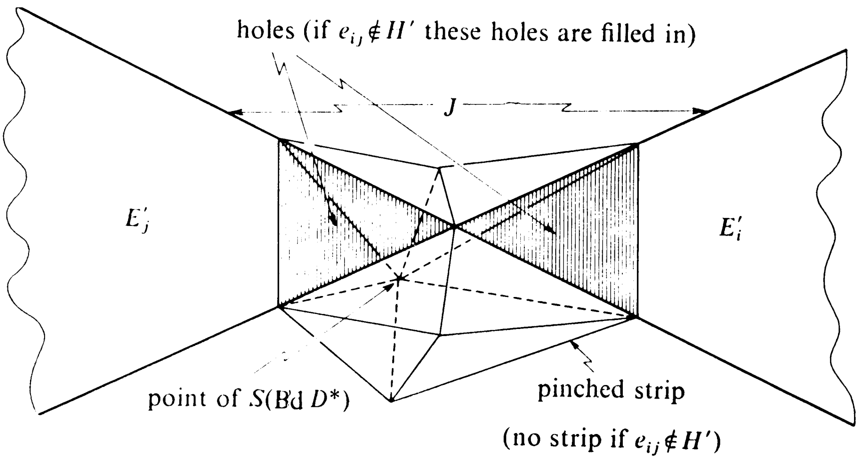


FIGURE 6

By this process we can obtain an s-disk E which is an admissible ε-alteration of D^* and such that $\mathrm{Bd}\, E$ is a conservatice ε-alteration of $\mathrm{Bd}\, D^*$. In fact, $|\mathrm{Bd}\, E| = |\mathrm{Bd}\, D^*|$ and thus $\mathrm{Bd}\, E$ is in B with only pinch point singularities.

Let G be an abstract 1-dim graph with $r - k$ vertices $v_{r+1}, \cdots, v_k$ and such that there is an edge e_{ij} connecting v_i to v_j if and only if a point p_{ij} on a double curve of $S(D^*)$, but not in the closed star (in $\beta^2 P$) of any triple, branch, or pinched branch point, such that $\mathrm{st}(p_{ij},\beta^2 P)\cap E'_i \neq \emptyset \neq \mathrm{st}(p_{ij},\beta^2 P)\cap E'_j$. Note that there will be at most one p_{ij} between a pair of triple, branch, or pinched branch points on a double curve of $S(D^*)$. Again III.11 will show that G is connected. Let G' be a maximal connected tree of G and make $\sum_{i=r+1}^{k}(E'_i)$ into a single nonsingular 2-sphere S by cutting out small holes in $\sum_{i=r+1}^{k}(E'_i)$ and connecting these holes by thin tubes near p_{ij} for those i,j such that $e_{ij}\in G'$. (We make these holes and tubes like illustration (a)(3) of the Addendum.) One more such tube from S to E near the middle of a double curve gives us an s-disk D'' satisfying the conclusion of III.5 if Λ is empty, except that some of the pinch point singularities may not be crossing points also. But III.8 applies and tells us there is an s-disk D' satisfying the conclusion of III.5 with $(B,M,\Lambda,D^*,\varepsilon)$. This contradicts our assumption that $(B,M,\Lambda,D^*,\varepsilon)\in\mathfrak{B}_1^*$. Note that this assumption is the only fact we used about D^* in Case 2.

Thus Case 1 and Case 2 together show

PROPOSITION III.15. $\mathfrak{B}_1^* = \emptyset$.

3c. In this section we use III.15 to show that $\mathfrak{B}^*$ is empty and therefore that III.5 is true.

Let $(B,M,\Lambda,D^*,\varepsilon)$ belong to $\mathfrak{B}^*$ and suppose $|D^*|$ to be a subcomplex of αM and $S(D^*)$ to be a subcomplex of $|D^*|$. A point in $S(\mathrm{Bd}\,D^*)$ is either (a) a branch point of D^*, or (b) a pinched branch point. Put small polyhedral balls $B_1,\cdots,B_k$ about each of the points of Type (a) and (b). The ball $B(v)$ about v is $\mathrm{st}(v,\alpha^n M)$, for n ($\geqq 2$) large enough that $B(v)\subset\varepsilon$-neighborhood in M of $S(\mathrm{Bd}\,D^*)$. One can then construct very thin tubes connecting the B_i's, such that $|\mathrm{Bd}\,D^*| - \sum B_i$ is contained on the union, T, of the tubes, and $|\mathrm{int}\,D^*| - \sum B_i$ is contained in one component of $M-(T+\sum B_i)$. Figure 7 is given in lieu of a precise description of T, because the author believes that "a picture is worth a thousand words" and that a precise description could easily take a thousand words. Note that, at points of $|\mathrm{Bd}\,D^*| - \sum B_i$, $|D^*|$ looks locally like the edge of a half-plane in E^3.

Let P be the closure of the component of $M-(T+\sum B_i)$ that contains $|\mathrm{int}\,D^*| - \sum B_i$. Let $T' = P\cap(T+\sum B_i)$ and let r be a retraction of $M-\mathrm{int}\,P$ onto $|\mathrm{Bd}\,D^*|$. If we make D^* simplicial, then the inverse image of a closed star of a vertex in $|D^*|$ is a closed star of a vertex in Δ. Therefore, $D^{*-1}(|D^*| - \mathrm{int}(\sum B_i))$ is a subdisk Δ' of Δ; and thus, $D^*\cap P \equiv D^*|\Delta'$ is an s-disk such that $(D^*\cap P,\ \mathrm{Bd}(D^*\cap P))$ is in rnp in $(P,\mathrm{Bd}\,P = T')$. It can be checked that $(T',P,(r|T')_*^{-1}\Lambda, D^*\cap P,\varepsilon')\in\mathfrak{B}_1$, where $\varepsilon'<\varepsilon$ is very small compared to the "thickness" of T'. Therefore, since $T' = \mathrm{Bd}\,P$ and $\mathfrak{B}_1^* = \emptyset$, there is a nonsingular s-disk D_1 which satisfies (1.1)–(1.4) and the Addendum of III.5 with $(T',P,(r|T')_*^{-1}\Lambda,\ D^*\cap P,\varepsilon)$. Also D_1 is a (conservative, if $\Lambda=\emptyset$) admissible ε-alteration of D^*. However, since $|\mathrm{Bd}\,D_1|\not\subset B$, $\mathrm{Bd}\,D_1$ is not an ε-alteration of $\mathrm{Bd}\,D^*$ in B, even though $|\mathrm{Bd}\,D_1| - \sum B_i = |\mathrm{Bd}\,D^*| - \sum B$.

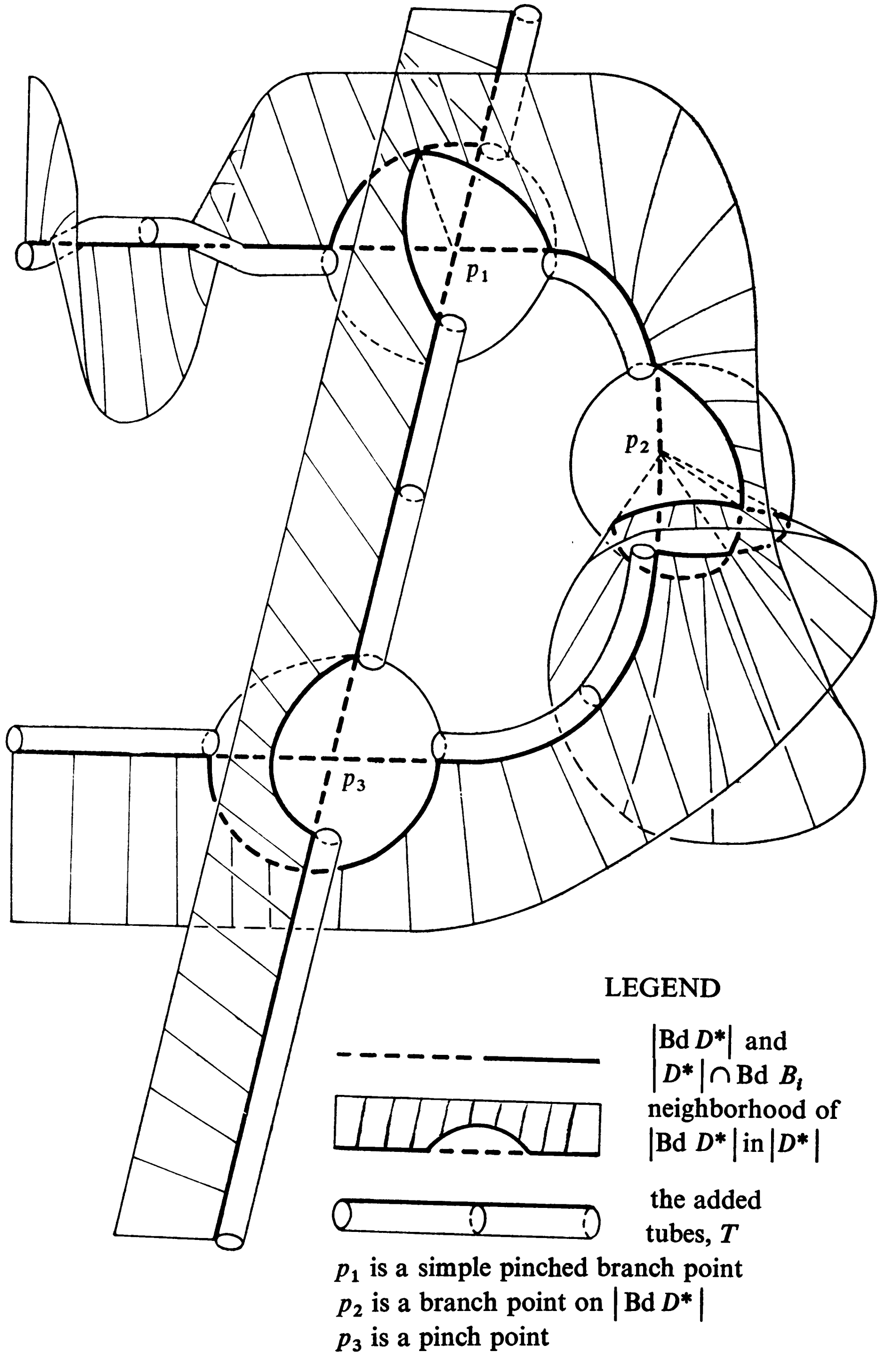

p_1
p_2
p_3
LEGEND
|Bd D^*| and
|D^*| ∩ Bd B_i
neighborhood of
|Bd D^*| in |D^*|
the added
tubes, T
p_1 is a simple pinched branch point
p_2 is a branch point on |Bd D^*|
p_3 is a pinch point

FIGURE 7

If $|\operatorname{Bd} D_1|$ intersects B_i, then $|\operatorname{Bd} D_1| \cap B_i$ is the union of one or two arcs. Also $|\operatorname{Bd} D_1| \cap B_i$ is a (conservative, if $\Lambda = \emptyset$) admissible ε'-alteration of $|D^*| \cap \operatorname{Bd} B_i$. We now forget about P and consider B. The triple $(B_i, B_i \cap B, B_i \cap |\operatorname{Bd} D_1|)$ is pwl homeomorphic under h to ($C^3, C^3 \cap xy$-plane, the disjoint union of one or two arcs on $\operatorname{Bd} C^3$ with end points on the xy-plane).

Let $A_1, (A_2)$ be the arc(s) on $\operatorname{Bd} C^3$. By "adding" to $D_1 h^{-1}$ (cone from the origin over $A_1(+A_2)$) for each ball B_i we get an s-disk D_2 which is nonsingular except for pinch points and which satisfies (1.1)–(1.3) and Addendum of III.5.

By appealing to III.8, we get an s-disk D' which satisfies the conclusion of III.5 with $(B, M, \Lambda, D^*, \varepsilon)$.

Thus in 3c we have shown that $\mathfrak{B}^* = \emptyset$ and the proof of III.5 is at last finished.

4. **Proof of III.2.** Since L is normal it is homeomorphic to a 1-dim complex. It follows from the two-dimensional Schoenflies Theorem that B can be triangulated so that some subdivision of $|L|$ is a subcomplex of the 1-skeleton of B. It follows from [1, Theorem 6] that M can be triangulated so that some subdivision of B is a subcomplex of M. Thus we may assume in III.2 that L is polyhedral in B and that B is a 2-submanifold of M.

Thus III.4 tells us that L is the boundary of a polyhedral s-disk D^* in M such that $|\operatorname{int} D^*| \cap |L| = \emptyset$.

By II.2 (D^*, L) may be assumed to be in rnp in (M, B). III.2 now follows from III.5.

5. **Proof of III.1.** There is a normal loop L' which is homotopic on B to L. Since $L'(M) = L(M) = 1$, L' bounds a singular disk D in M. But $\operatorname{int} D$ may be pushed off B without moving L', thus L' is self-unlinked and III.1 follows from III.2.

IV. Dehn's Lemma

1. The following theorem was first proved by C. D. Papakyriakopoulos [12] and later a shorter proof using 2-sheeted coverings was given by A. Shapiro and J. H. C. Whitehead [14].

Dehn's Lemma. *If D is a polyhedral s-disk in a 3-manifold M, such that $S(D) \cap \operatorname{Bd} D = \emptyset$, then there is a nonsingular disk D' in M, such that* $\operatorname{Bd} D' = \operatorname{Bd} D$.

In the same paper Shapiro and Whitehead extended Dehn's Lemma to a theorem concerning disks with n holes [14, Theorem (1.1)]. R. H. Bing has pointed out [3, p. 10] that by applying the polyhedral approximation of surface theorems of [2; 6] one can remove the requirement that D be polyhedral. Bing has also asked [3, p. 10] whether the requirement "$S(D) \cap \operatorname{Bd} D = \emptyset$" could be replaced by "$S(\operatorname{Bd} D) = \emptyset$," or, stated another way:

QUESTION IV.1. *Does every self-unlinked simple closed curve in a* 3-*manifold* M *bound a* (*nonsingular*) *disk in* M?

The following theorem gives a partial answer to IV.1.

THEOREM IV.2. *Every self-unlinked tame simple closed curve in a* 3-*manifold* M *bounds a* (*nonsingular*) *disk in* M.

Other partial answers to IV.1 are included in [8].

For the case that M has no boundary, Theorem IV.2 has been proved, but not published, by R. H. Bing; and, according to a note in [6, p. 889], C. D. Papakyriakopoulos also has a proof. The proof included here is based on III.4 and III.5.

Part IV will be concluded with a proof of the following strengthening of the polyhedral Dehn's Lemma:

THEOREM IV.3. *Let* M *be a* 3-*manifold and* D *a polyhedral* s-*disk in* M. ***If*** $S(\mathrm{Bd}\,D)=\varnothing$ *and* ε *is a positive number*, ***then*** *there is a nonsingular* s-*disk* D' *in* M, *such that* $\mathrm{Bd}\,D'=\mathrm{Bd}\,D$ *and* D' *is a conservative* ε-*alteration of* D.

2. **Proof of IV. 2.** Let J be a self-unlinked tame simple closed curve in a 3-manifold M. Since J is tame we may assume without loss of generality that J is polyhedral. By III.4 J bounds a polygonal s-disk D such that $|J|\cap|D-J|=\varnothing$. Let N be any 2-submanifold of M that contains J. (Since J is polygonal such an N exists, but, since $S(J)=\varnothing$, N has no effect on the proof.) By II.2 we may assume that (D,J) is in rnp in (M,N). By III.5 there is an s-disk D' which is nonsingular except for crossing pinch points and such that $(D',\mathrm{Bd}\,D')$ is a conservative ε-alteration of (D,J). But $S(J)=\varnothing$; therefore $|\mathrm{Bd}\,D'|=|J|$ and $S(\mathrm{Bd}\,D')=\varnothing$. (See I.14.) Thus $\mathrm{Bd}\,D'=J$ which finishes the proof of IV.2.

3. **Proof of IV. 3.** By II. 2 there is an s-disk D^* in M such that $\mathrm{Bd}\,D^*=\mathrm{Bd}\,D$, $(D^*,\mathrm{Bd}\,D^*)$ is in rnp in M (see last sentence of I.14), and D^* is a conservative $\varepsilon/2$-alternation of D.

By III.5 there is an s-disk D' in M such that $(D',\mathrm{Bd}\,D')$ is a conservative $\varepsilon/2$-alteration of $(D^*,\mathrm{Bd}\,D^*)$. But $S(\mathrm{Bd}\,D^*)=\varnothing$, therefore $|\mathrm{Bd}\,D'|=|\mathrm{Bd}\,D^*|$ and $S(\mathrm{Bd}\,D')=\varnothing$. We conclude that $\mathrm{Bd}\,D'=\mathrm{Bd}\,D^*=\mathrm{Bd}\,D$ and that, since the only possible singularities of D' were pinch points on $\mathrm{Bd}\,D'$, D' must be nonsingular. This concludes the proof of IV.3.

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